Reflection from Fractal Cantor Layers in a Rectangular Waveguide

T. J. Cui and C. H. Liang, *Senior Member, IEEE*

***Abstract*—The multilayered medium modeled by a Cantor bar is a type of fractal structure that has found wide applications in some optical and quantum areas. The reflection properties of wave in a rectangular waveguide filled with fractal Cantor bar layers are investigated. By introducing a concept of self-similarity of networks, a novel exact self-similar algorithm for reflection and transmission coefficients is derived. Numerical examples show that the reflection from the fractal layer in a waveguide has some special properties.**

I. Introduction

SINCE the fractal concept was first proposed by Mandelbrot in 1970s [1], its wide applications have been found in natures and social sciences. In the optical and electromagnetic areas, a lot of researches have been made by Jakeman, Bourrely, Jaggard and other scholars [3] ~ [8]. One of the above applications is the problem of electromagnetic wave interactions with finely divided layers characterized by a specified fractal distribution, which finds applications in areas as varied as multilayer synthesis and analysis, distributed-feed back integrated-optical structures, and multiple-quantum-well devices [7], [8]. In this letter, we will consider the problem of guided wave interactions with fractal layers, in which the reflection properties from the fractal Cantor bar layers in a rectangular waveguide are investigated and a novel exact self-similar algorithm for reflection and transmission coefficients is derived by introducing the concept of self-similarity of networks.

II. Rectangular Waveguide Filled with Fractal Cantor Bar Layers and Self-Similar Networks

Fractals are characterized by their dilation symmetry or property of self-similarity in which the object is invariant under a change of scale and displacement. Cantor set is a typical example of the fractals. As demonstrated in Fig. 1, a Cantor bar layers in a rectangular waveguide is generated from a bar of unit length by repeatedly removing the middle $1/R(R>1)$ of each existing bar. Each part of the bar layers at a given stage of growth, when magnified, appears as the set in a previous stage. Cantor dust is formed when the height of the bars becomes vanishingly small.

If the height of the bar in growth stage 0 is Δ, the height of each bar of the Cantor bars in growth stage m is easily obtained from the generation of the fractal Cantor layers:

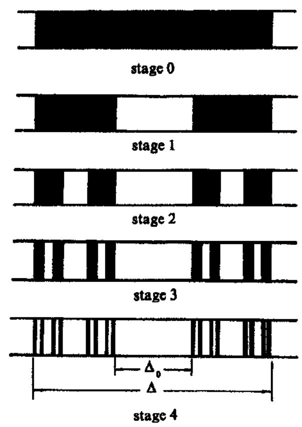

Fig. 1. Generation of a fractal Cantor bar ($R = 3$) in a rectangular waveguide and its stages of growth.

$$d_m = \left(\frac{R-1}{2R}\right)^m \Delta, \qquad m = 0, 1, 2, \cdots, \tag{1}$$

and the relation between Δ_0 (the distance of the two bars in growth stage 1, see Fig. 1) and d_m is further derived

$$\Delta_0 = \frac{\Delta}{R} = \frac{(2R)^m}{R(R-1)^m} d_m, \qquad m = 1, 2, 3, \cdots. \tag{2}$$

Because the total number of the bars in growth stage m is $M(m) = 2^m$, the fractal dimension D_c of the Cantor dust is achieved by using the usual definition of fractals [1]

$$D_c = -\lim_{m\to\infty} \frac{\ln M(m)}{\ln(d_m)} = \frac{\ln 2}{\ln(2R) - \ln(R-1)}, \tag{3}$$

which is just the Hausdorff's dimension [2].

As we can see from Fig. 1, this fractal dimension indicates that the Cantor dust occupies more space than a point ($D_c \to 0$) but less than a line ($D_c \to 1$). As the Cantor bar ideally approaches higher stages of growth, its total length approaches zero, and the number of bars becomes unbounded.

According to microwave network theory, the Cantor set in Fig. 1 will be equivalent to multiple cascaded networks which we call Cantor cascaded networks. These Cantor cascaded networks are characterized by their dilation symmetry, thus we define them as self-similar networks.

Manuscript received March 16, 1993. This work was supported in part by the National Doctoral Funds.

The authors are with the Department of Electromagnetic Field Engineering, Xidian University, Xi'an, 710071, People's Republic of China.

IEEE Log Number 9211460.

1051-8207/93$03.00

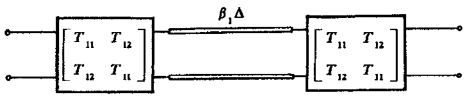

Fig. 2. The equivalent microwave network of the fractal Cantor bar in growth stage 0.

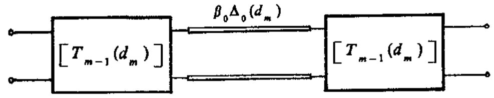

Fig. 3. The equivalent microwave network of the fractal Cantor bars in growth stage m.

III. Reflection and Transmission Coefficients of the Fractal Cantor Layers in a Rectangular Waveguide

To obtain the self-similar networks of the fractal Cantor layers in a rectangular waveguide, we first consider the mode propagation constant β and characteristic wave impedance Z_0 of the waveguide filled with even dielectrics. Suppose that the size of the waveguide is $a \times b$ and the permittivity and permeability relative to free space of the dielectrics are ε_r and μ_r respectively, by the guided wave theory we have [9]

$$\beta(\mu_r, \varepsilon_r) = 2\pi\gamma f\sqrt{\mu_r\varepsilon_r}/c \tag{4}$$

$$Z_0^{\mathrm{TE}}(\mu_r, \varepsilon_r) = \eta_0\sqrt{\mu_r/\varepsilon_r}/\gamma,$$
$$Z_0^{\mathrm{TM}}(\mu_r, \varepsilon_r) = \eta_0\gamma\sqrt{\mu_r/\varepsilon_r}, \tag{5}$$

where Z_0^{TE} and Z_0^{TM} represent the characteristic wave impedances of TE_{pq} mode and TM_{pq} mode respectively, and

$$\gamma = \sqrt{1-(f_c/f)^2},$$
$$f_c = c\sqrt{(p/a)^2+(q/b)^2}/(2\sqrt{\mu_r\varepsilon_r}),$$
$$p = 0, 1, 2, \cdots; \quad q = 0, 1, 2, \cdots; \quad p = q = 0 \text{ excepted},$$

in which f is the frequency; $c = 3 \times 10^8$ m/s is the light speed in free space.

For the fractal Cantor bar layers in the rectangular waveguide shown in Fig. 1, the white areas represent the background or host dielectric, whose permittivity and permeability relative to free space are denoted ε_e and μ_e; while the shaded areas represent the embedded-layer, denoted ε_r and μ_r. Thus the equivalent microwave network of the Cantor bar in growth stage 0 is obtained, as shown in Fig. 2, where T_{11} and T_{12} are the T parameters of the transmission line junction. By the network theory, we have

$$T_{11} = 1/\sqrt{1-R_{01}^2}, \qquad T_{12} = R_{01}/\sqrt{1-R_{01}^2}, \tag{6}$$

where

$$R_{01} = (Z_{01} - Z_{00})/(Z_{01} + Z_{00}), \tag{7}$$

in which $Z_{01} = Z_0(\mu_r, \varepsilon_r)$, $Z_{00} = Z_0(\mu_e, \varepsilon_e)$, $\beta_1 = \beta(\mu_r, \varepsilon_r)$, as expressed in (4) and (5).

According to the properties of T parameters, the total T parameters of the Cantor bar in growth stage 0, $[T_0] = \begin{bmatrix} u_0 & v_0 \\ -v_0 & w_0 \end{bmatrix}$ will be achieved from Fig. 2,

$$u_0(d_0) = [\exp(j\beta_1 d_0) - R_{01}^2 \exp(-j\beta_1 d_0)]/(1-R_{01}^2) \tag{8a}$$
$$v_0(d_0) = -j2R_{01}\sin(\beta_1 d_0)/(1-R_{01}^2) \tag{8b}$$
$$w_0(d_0) = [\exp(-j\beta_1 d_0) - R_{01}^2 \exp(j\beta_1 d_0)]/(1-R_{01}^2) \tag{8c}$$

and $\det[T_0] = u_0 w_0 + v_0^2 = 1$ is satisfied.

For the fractal Cantor bars in growth stage 1 (see Fig. 1), their equivalent microwave network should consist of two sub-networks and a section of transmission line whose propagation constant is $\beta_0 = \beta(\mu_e, \varepsilon_e)$ and length is Δ_0, which are cascaded. Each of the sub-networks is the same as the total network of the Cantor bar in growth stage 0, if d_0 is replaced by d_1. This is just the self-similarity of networks. On the analogy of above idea, the equivalent microwave network of the Cantor bars in growth stage m also consists of two sub-networks and a section of transmission line. Each sub-networks is the same as the total network of the bars in growth stage $m-1$ if d_{m-1} is replaced by d_m, as shown in Fig. 3.

By using the properties of T parameters, the total T parameters of the Cantor bars in growth stage m, $[T_m] = \begin{bmatrix} u_m & v_m \\ -v_m & w_m \end{bmatrix}$, can be derived from Fig. 3

$$u_m(d_m) = u_{m-1}^2(d_m)\exp[j\theta_m(d_m)] - v_{m-1}^2(d_m)\exp[-j\theta_m(d_m)] \tag{9a}$$
$$v_m(d_m) = v_{m-1}(d_m)\{u_{m-1}(d_m)\exp[j\theta_m(d_m)] + w_{m-1}(d_m)\exp[-j\theta_m(d_m)]\} \tag{9b}$$
$$w_m(d_m) = w_{m-1}^2(d_m)\exp[-j\theta_m(d_m)] - v_{m-1}^2(d_m)\exp[j\theta_m(d_m)] \tag{9c}$$

and $\det[T_m] = \det[T_{m-1}] = 1$ is satisfied, where

$$\theta_m(d_m) = \beta_0\Delta_0 = \beta_0\frac{(2R)^m}{R(R-1)^m}d_m. \tag{10}$$

Equation (9) defines an iterative scheme. If the reflection and transmission coefficients of a set of Cantor bar fractal layers of growth stage n are to be computed, iterating equation (9) is used with initial values $u_0(d_n), v_0(d_n)$, and $w_0(d_n)$; $m = 1, 2, \cdots, n$. From the relationship between T parameters and S scattering parameters, we have

$$S_{11}(f) = S_{22}(f) = -v_n(d_n)/u_n(d_n) \tag{11}$$
$$S_{12}(f) = S_{21}(f) = 1/u_n(d_n), \tag{12}$$

which are just the reflection and transmission coefficients of the Cantor bars in growth stage n.

From these deductions, the exact iterating algorithm of (9)–(10) dramatically reduces the amount of calculation needed to compute the reflection and transmission coefficients of the discrete fractal layers. For a set of Cantor bars of growth stage n, for example, only n iterations are needed. To solve the same problem with the chain-matrix approach, the multiplication of $(2^{n+1}-1)$ matrices is required.

The next, we consider the case of $n \to \infty$. Suppose that the limits of u_n, v_n and w_n exist, and let $u = \lim_{n\to\infty} u_n$, $v = \lim_{n\to\infty} v_n$, $w = \lim_{n\to\infty} w_n$, we will obtain from (9)

$$w = (1 - ue^{j\theta})e^{j\theta}, u = u^2e^{j\theta} - v^2e^{-j\theta}, \tag{13}$$

where, $\theta = \beta_0\Delta/R$. Considering the condition: $\det[T] = uw + v^2 = 1$, (14) gives

$$e^{-j\theta} = 0, \qquad (14)$$

which is a contradictory equation. Thus the supposition is not true. That is to say, the limit of the reflection (or transmission) coefficient of the Cantor dust ($n \to \infty$) do not exist.

IV. Numerical Examples and Discussion

It is well known that the basic propagation mode of a rectangular waveguide is TE_{10} mode. We have computed the reflection coefficients of two typical waveguides (BJ-32 and BJ-100) filled with fractal Cantor layers when TE_{10} mode is propagated, where the structural parameters are chosen as $\varepsilon_r = 4, \mu_r = 1, \varepsilon_e = 1, \mu_e = 1, \Delta = 2a$, and the range of frequency is $f \in [f_c, 2f_c]$. Fig. 4(a) gives the the reflection properties of the Cantor bars ($R = 3$) of growth stage 0, 2, 3, 4, 8, 10 in the BJ-32 waveguide; Fig. 4(b) gives those of the Cantor bars ($R = 2$) of growth stage 0, 2, 3, 4, 5, 6 in the BJ-0 waveguide.

From Fig. 4, the waveguides filled with fractal Cantor bar layers have better band-pass or band-elimination behaviors than those filled with even dielectrics (i.e., $n = 0$). For example, in BJ-32 waveguide, well frequency-select property exists when $n = 2$, well band-elimination property exists when $n = 3$ and well band-pass property exists when $n = 4$. Similar conclusion can be made in BJ-100 waveguide. Thus, some special behaviors will be obtained if the parameters of the fractal Cantor bar layers filled in the waveguide are carefully chosen.

On the other hand, the method presented here and the concept of self-similarity of networks can be used in various types of computation involving self-similar fractal structure. For example, the problem mentioned in [8] can also be solved by our method. Comparing the computed results, they are completely the same, which shows the availability of this scheme.

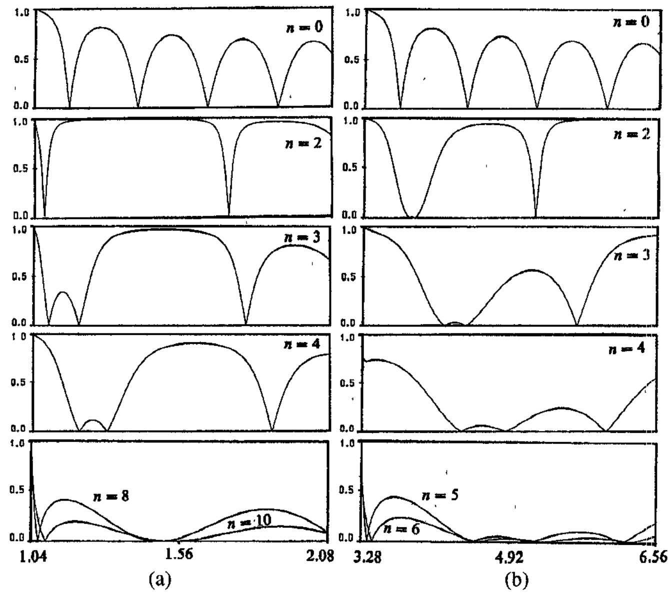

Fig. 4. The reflection coefficients of the fractal Cantor layers in the rectangular waveguides. (a) BJ-32 waveguide (72.14 × 34.04 mm^2) and $R = 3$; (b) BJ-100 waveguide (22.86 × 10.16 mm^2) and $R = 2$.

References

[1] B. B. Mandelbrot, *The Fractal Geometry of Nature*. San Francisco, CA: Freeman, 1982, ch. 1.
[2] K. Falconer, *Fractal Geometry*. New York: John Wiley, Chichester, 1990, ch. 1.
[3] E. Jakeman, "Fresnel scattering by a corrugated random surface with fractal slope," *J. Opt. Soc. Am.*, vol. 72, pp. 1034–1040, 1982.
[4] C. Bourrely, P. Chiappetta, and B. Torresani, "Light scattering by particles of arbitrary shape: A fractal approach," *J. Opt. Soc. Am.*, vol. A-3, pp. 250–255, 1986.
[5] D. L. Jaggard and Y. Kim, "Diffraction by bandlimited fractal screens," *J. Opt. Soc. Am.*, vol. A-4, pp. 1055–1061, 1987.
[6] D. L. Jaggard and X. Sun, "Scattering from fractally corrugated surfaces," *J. Opt. Soc. Am.*, vol. A-7, pp. 1131–1136, 1990.
[7] ——, "Scattering from bandlimited fractal fibers," *IEEE Trans. Antennas Propagat.*, vol. 37, pp. 1591–1596, 1989.
[8] ——, "Reflection from fractal multilayers," *Optics Lett.*, vol. 15, pp. 1428–1430, 1990.
[9] R. F. Harrington, *Time-Harmonic Electromagnetic Field*. New York: McGraw-Hill, 1961, pp. 148–152.